Associated special functions and coherent states

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Abstract: A hypergeometric type equation satisfying certain conditions defines either a finite or an infinite system of orthogonal polynomials. We present in a unified and explicit way all these systems of orthogonal polynomials, the associated special functions and some systems of coherent states. This general formalism allows us to extend some results known only in particular cases.

Key-Words: Orthogonal polynomials, Associated special functions, Coherent states, Raising and lowering operators, Creation and annihilation operators, Hypergeometric-type equations.

1 Introduction

Many problems in quantum mechanics and mathematical physics lead to equations of the type

$$\sigma(s)y''(s)+\tau(s)y'(s)+\lambda y(s)=0 \qquad (1)$$

where $\sigma(s)$ and $\tau(s)$ are polynomials of at most second and first degree, respectively, and λ is a constant. These equations are usually called *equations of hypergeometric type* [15], and each can be reduced to the self-adjoint form

$$[\sigma(s)\varrho(s)y'(s)]'+\lambda\varrho(s)y(s)=0 \qquad (2)$$

by choosing a function ϱ such that

$$[\sigma(s)\varrho(s)]'=\tau(s)\varrho(s). \qquad (3)$$

The equation (1) is usually considered on an interval (a,b), chosen such that

$$\begin{array}{rll} \sigma(s)>0 & \text{for all} & s\in(a,b)\\ \varrho(s)>0 & \text{for all} & s\in(a,b) \\ \lim_{s\to a}\sigma(s)\varrho(s)=\lim_{s\to b}\sigma(s)\varrho(s)=0. \end{array} \qquad (4)$$

Since the form of the equation (1) is invariant under a change of variable $s\mapsto cs+d$, it is sufficient to analyse the cases presented in table 1. Some restrictions must be imposed on α, β in order for the interval (a,b) to exist. We prove that equation (1) defines an infinite sequence of orthogonal polynomials

$$\Phi_0,\ \ \Phi_1,\ \ \Phi_2,\ ...$$

in the case $\sigma(s)\in\{1,\ s,\ 1-s^2\}$, and a finite one

$$\Phi_0,\ \ \Phi_1,\ \ ...,\ \ \Phi_L$$

in the case $\sigma(s)\in\{s^2-1,\ s^2,\ s^2+1\}$.

The literature discussing special function theory and its application to mathematical and theoretical physics is vast, and there are a multitude of different conventions concerning the definition of functions. A unified approach is not possible without a unified definition for the associated special functions. In this paper we define them as

$$\Phi_{l,m}(s)=\left(\sqrt{\sigma(s)}\right)^m\frac{\mathrm{d}^m}{\mathrm{d}s^m}\Phi_l(s) \qquad (5)$$

where Φ_l are the orthogonal polynomials defined by equation (1). The table 1 allows one to pass in each case from our parameters α, β to the parameters used in different approach.

$\sigma(s)$	$\varrho(s)$	α,β	(a,b)
1	$e^{\frac{1}{2}\alpha s^2+\beta s}$	$\alpha<0$	$\mathbb{R}$
s	$s^{\beta-1}e^{\alpha s}$	$\alpha<0$ $\beta>0$	$(0,\infty)$
$1-s^2$	$(1+s)^{-\frac{\alpha-\beta}{2}-1}\times$ $\times(1-s)^{-\frac{\alpha+\beta}{2}-1}$	$\alpha<\beta$ $\alpha+\beta<0$	$(-1,1)$
s^2-1	$(s+1)^{\frac{\alpha-\beta}{2}-1}\times$ $\times(s-1)^{\frac{\alpha+\beta}{2}-1}$	$0<\alpha+\beta$ $\alpha<0$	$(1,\infty)$
s^2	$s^{\alpha-2}e^{-\frac{\beta}{s}}$	$\alpha<0$ $\beta>0$	$(0,\infty)$
s^2+1	$(1+s^2)^{\frac{1}{2}\alpha-1}\times$ $\times\, e^{\beta\arctan s}$	$\alpha<0$	$\mathbb{R}$

Table 1: Particular cases (in each case $\tau(s)=\alpha s+\beta$).

In our previous papers [6, 7], we presented a systematic study of the Schrödinger equations exactly solvable in terms of associated special functions following Lorente [14], Jafarizadeh and Fakhri [12]. In the present paper, our aim is to extend this unified formalism by including a larger class of creation/annihilation operators and some temporally stable coherent states of Gazeau-Klauder type [1, 8–10, 13].

2 Orthogonal polynomials of hypergeometric-type

Let $\tau(s)=\alpha s+\beta$ be a fixed polynomial, and let

$$\lambda_l=-\frac{\sigma''(s)}{2}l(l-1)-\tau'(s)l=-\frac{\sigma''}{2}l(l-1)-\alpha\, l \tag{6}$$

for any $l\in\mathbb{N}$. It is well-known [15] that for $\lambda=\lambda_l$, the equation (1) admits a polynomial solution $\Phi_l=\Phi_l^{(\alpha,\beta)}$ of at most l degree

$$\sigma(s)\Phi_l''+\tau(s)\Phi_l'+\lambda_l\Phi_l=0. \tag{7}$$

If the degree of the polynomial Φ_l is l then it satisfies the Rodrigues formula [15]

$$\Phi_l(s)=\frac{B_l}{\varrho(s)}\frac{d^l}{ds^l}[\sigma^l(s)\varrho(s)] \tag{8}$$

where B_l is a constant. Based on the relation

$$\{\ \delta\in\mathbb{R}\ |\ \lim_{s\rightarrow a}\sigma(s)\varrho(s)s^\delta=\lim_{s\rightarrow b}\sigma(s)\varrho(s)s^\delta=0\ \}$$

$$=\left\{\begin{array}{lll}[0,\infty) & \text{if} & \sigma(s)\in\{1,\ s,\ 1-s^2\}\\ [0,-\alpha) & \text{if} & \sigma(s)\in\{s^2-1,\ s^2,\ s^2+1\}\end{array}\right.$$

one can prove [7] that the system of polynomials $\{\Phi_l\ |\ l<\Lambda\}$, where

$$\Lambda=\left\{\begin{array}{lll}\infty & \text{for} & \sigma(s)\in\{1,\ s,\ 1-s^2\}\\ \frac{1-\alpha}{2} & \text{for} & \sigma(s)\in\{s^2-1,\ s^2,\ s^2+1\}\end{array}\right. \tag{9}$$

is orthogonal with weight function $\varrho(s)$ in (a,b). This means that equation (1) defines an infinite sequence of orthogonal polynomials

$$\Phi_0,\quad \Phi_1,\quad \Phi_2,\ ...$$

in the case $\sigma(s)\in\{1,\ s,\ 1-s^2\}$, and a finite one

$$\Phi_0,\quad \Phi_1,\quad ...,\quad \Phi_L$$

with $L=\max\{l\in\mathbb{N}\ |\ l<(1-\alpha)/2\}$ in the case $\sigma(s)\in\{s^2-1,\ s^2,\ s^2+1\}$.

The polynomials $\Phi_l^{(\alpha,\beta)}$ can be expressed in terms of the classical orthogonal polynomials as

$$\Phi_l^{(\alpha,\beta)}(s)=$$

$$\left\{\begin{array}{lll}H_l\left(\sqrt{\frac{-\alpha}{2}}\,s-\frac{\beta}{\sqrt{-2\alpha}}\right) & \text{if} & \sigma(s)=1\\ L_l^{\beta-1}(-\alpha s) & \text{if} & \sigma(s)=s\\ P_l^{(-\frac{\alpha+\beta}{2}-1,\ \frac{-\alpha+\beta}{2}-1)}(s) & \text{if} & \sigma(s)=1-s^2\\ P_l^{(\frac{\alpha-\beta}{2}-1,\ \frac{\alpha+\beta}{2}-1)}(-s) & \text{if} & \sigma(s)=s^2-1\\ \left(\frac{s}{\beta}\right)^l L_l^{1-\alpha-2l}\left(\frac{\beta}{s}\right) & \text{if} & \sigma(s)=s^2\\ i^l P_l^{(\frac{\alpha+i\beta}{2}-1,\ \frac{\alpha-i\beta}{2}-1)}(is) & \text{if} & \sigma(s)=s^2+1\end{array}\right. \tag{10}$$

where H_n, L_n^p and $P_n^{(p,q)}$ are the Hermite, Laguerre and Jacobi polynomials, respectively.

3 Associated special functions, raising and lowering operators

Let $l\in\mathbb{N}$, $l<\Lambda$, and let $m\in\{0,1,...,l\}$. The functions

$$\Phi_{l,m}(s)=\kappa^m(s)\frac{d^m}{ds^m}\Phi_l(s) \tag{11}$$

where

$$\kappa(s) = \sqrt{\sigma(s)}$$

are called the *associated special functions*. If we differentiate (7) m times and then multiply the obtained relation by $\kappa^m(s)$ then we get the equation

$$H_m \Phi_{l,m} = \lambda_l \Phi_{l,m} \tag{12}$$

where H_m is the differential operator

$$H_m = -\sigma(s)\frac{d^2}{ds^2} - \tau(s)\frac{d}{ds} + \frac{m(m-2)}{4}\frac{(\sigma'(s))^2}{\sigma(s)}$$

$$+\frac{m\tau(s)}{2}\frac{\sigma'(s)}{\sigma(s)} - \frac{1}{2}m(m-2)\sigma''(s) - m\tau'(s). \tag{13}$$

The relation

$$\langle f, g\rangle = \int_a^b \overline{f(s)}\, g(s)\varrho(s)ds \tag{14}$$

defines a scalar product on the space

$$\mathcal{H}_m = \mathrm{span}\{\Phi_{l,m} \mid m \leq l < \Lambda\}$$

spanned by $\{\Phi_{l,m} \mid m \leq l < \Lambda\}$. For each $m < \Lambda$, the special functions $\Phi_{l,m}$ with $m \leq l < \Lambda$ are orthogonal with weight function $\varrho(s)$ in (a, b), and the functions corresponding to consecutive values of m are related through the raising/lowering operators [6, 7, 12]

$$\begin{array}{l} A_m = \kappa(s)\frac{d}{ds} - m\kappa'(s) \\ A_m^+ = -\kappa(s)\frac{d}{ds} - \frac{\tau(s)}{\kappa(s)} - (m-1)\kappa'(s) \end{array} \tag{15}$$

namely,

$$\begin{array}{l} A_m\Phi_{l,m} = \left\{ \begin{array}{lll} 0 & \text{for} & l = m \\ \Phi_{l,m+1} & \text{for} & m < l < \Lambda \end{array} \right. \\ A_m^+\Phi_{l,m+1} = (\lambda_l - \lambda_m)\Phi_{l,m} \quad \text{for} \quad 0 \leq m < l < \Lambda. \end{array} \tag{16}$$

In addition, we have the relations [5, 11]

$$\Phi_{l,m} = \frac{A_m^+}{\lambda_l - \lambda_m}\frac{A_{m+1}^+}{\lambda_l - \lambda_{m+1}} \cdots \frac{A_{l-1}^+}{\lambda_l - \lambda_{l-1}}\Phi_{l,l} \tag{17}$$

for $0 \leq m < l < \Lambda$, and

$$H_m - \lambda_m = A_m^+ A_m \qquad H_{m+1} - \lambda_m = A_m A_m^+ \tag{18}$$

$$H_m A_m^+ = A_m^+ H_{m+1} \qquad A_m H_m = H_{m+1} A_m \tag{19}$$

for $m + 1 < \Lambda$.

The functions

$$\phi_{l,m} = \Phi_{l,m}/||\Phi_{l,m}|| \tag{20}$$

where

$$||f|| = \sqrt{\langle f, f\rangle} \tag{21}$$

are the *normalized associated special functions*. Since [6, 7]

$$||\Phi_{l,m+1}|| = \sqrt{\lambda_l - \lambda_m}\, ||\Phi_{l,m}|| \tag{22}$$

they satisfy the relations

$$\begin{array}{l} A_m\phi_{l,m} = \left\{ \begin{array}{lll} 0 & \text{for} & l = m \\ \sqrt{\lambda_l - \lambda_m}\phi_{l,m+1} & \text{for} & m < l < \Lambda \end{array} \right. \\ A_m^+\phi_{l,m+1} = \sqrt{\lambda_l - \lambda_m}\phi_{l,m} \quad \text{for } 0 \leq m < l < \Lambda \\ \phi_{l,m} = \frac{A_m^+}{\sqrt{\lambda_l - \lambda_m}}\frac{A_{m+1}^+}{\sqrt{\lambda_l - \lambda_{m+1}}} \cdots \frac{A_{l-1}^+}{\sqrt{\lambda_l - \lambda_{l-1}}}\phi_{l,l}. \end{array} \tag{23}$$

4 Coherent states in the case $\sigma(s) \in \{1,\ s,\ 1 - s^2\}$

Let m be a fixed natural number and γ a fixed real number. The sequence

$$\phi_{m,m}, \quad \phi_{m+1,m}, \quad \phi_{m+2,m}, \ \ldots$$

is a complete orthonormal sequence in the Hilbert space

$$\mathcal{H} = \left\{ \varphi : (a, b) \longrightarrow \mathbb{C} \;\middle|\; \int_a^b |\varphi(s)|^2 \varrho(s)\, ds < \infty \right\}$$

with scalar product (14), for any $m \in \mathbb{N}$.

4.1 Creation and annihilation operators

The linear operators (see figure 1)

$$\begin{array}{l} a_m,\ a_m^+ : \mathcal{H}_m \longrightarrow \mathcal{H}_m \\ \qquad a_m = U_m^{-1}A_m \qquad a_m^+ = A_m^+ U_m \end{array} \tag{24}$$

defined by using the unitary operator

$$\begin{array}{l} U_m : \mathcal{H}_m \longrightarrow \mathcal{H}_{m+1} \\ \qquad U_m\phi_{l,m} = \mathrm{e}^{-\mathrm{i}\gamma(\lambda_{l+1} - \lambda_l)}\phi_{l+1,m+1} \end{array} \tag{25}$$

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots \\
\phi_{3,0} \arrow[r, shift right, "A_0"'] \arrow[d, shift left, "a_0"] & \phi_{3,1} \arrow[l, shift right, "A_0^+"'] \arrow[r, shift right, "A_1"'] \arrow[d, shift left, "a_1"] & \phi_{3,2} \arrow[l, shift right, "A_1^+"'] \arrow[r, shift right, "A_2"'] \arrow[d, shift left, "a_2"] & \phi_{3,3} \arrow[l, shift right, "A_2^+"'] \\
\phi_{2,0} \arrow[u, shift left, "a_0^+"] \arrow[ur, "U_0"] \arrow[r, shift right, "A_0"'] \arrow[d, shift left, "a_0"] & \phi_{2,1} \arrow[u, shift left, "a_1^+"] \arrow[ur, "U_1"] \arrow[l, shift right, "A_0^+"'] \arrow[r, shift right, "A_1"'] \arrow[d, shift left, "a_1"] & \phi_{2,2} \arrow[u, shift left, "a_2^+"] \arrow[ur, "U_2"] \arrow[l, shift right, "A_1^+"'] & \\
\phi_{1,0} \arrow[u, shift left, "a_0^+"] \arrow[ur, "U_0"] \arrow[r, shift right, "A_0"'] \arrow[d, shift left, "a_0"] & \phi_{1,1} \arrow[u, shift left, "a_1^+"] \arrow[ur, "U_1"] \arrow[l, shift right, "A_0^+"'] & & \\
\phi_{0,0} \arrow[u, shift left, "a_0^+"] \arrow[ur, "U_0"] & & &
\end{tikzcd}
$$

Fig. 1: The operators A_m, A_m^+, a_m, a_m^+ and U_m relating the functions $\phi_{l,m}$.

are mutually adjoint,

$$
\begin{aligned}
a_m\phi_{l,m} &= \begin{cases} 0 & \text{for } l=m \\ \sqrt{\lambda_l-\lambda_m}\,\mathrm{e}^{\mathrm{i}\gamma(\lambda_l-\lambda_{l-1})}\phi_{l-1,m} & \text{for } l>m \end{cases} \\
a_m^+\phi_{l,m} &= \sqrt{\lambda_{l+1}-\lambda_m}\,\mathrm{e}^{-\mathrm{i}\gamma(\lambda_{l+1}-\lambda_l)}\phi_{l+1,m} \quad \text{for } l\geq m
\end{aligned} \tag{26}
$$

and

$$H_m-\lambda_m = a_m^+a_m \tag{27}$$

$$
\begin{aligned}
[a_m^+,a_m]\phi_{l,m} &= (\lambda_l-\lambda_{l+1})\phi_{l,m} \\
&= (\sigma'' l+\alpha)\phi_{l,m}.
\end{aligned} \tag{28}
$$

Since the operator $R_m=[a_m^+,a_m]$ satisfies the relations

$$[R_m,a_m^+]=\sigma'' a_m^+ \qquad [R_m,a_m]=-\sigma'' a_m \tag{29}$$

the Lie algebra $\mathbb{L}_m$ generated by $\{a_m^+,a_m\}$ is finite dimensional.

Theorem 1. *The Lie algebra $\mathbb{L}_m$ is isomorphic*

$$\textit{to} \quad \begin{cases} h(2) & \textit{if} \quad \sigma(s)\in\{1,\ s\} \\ su(1,1) & \textit{if} \quad \sigma(s)=1-s^2. \end{cases}$$

Proof. In the case $\sigma(s)\in\{1,s\}$ the operator R_m is a constant operator, namely, $R_m=\alpha$. Since $\alpha<0$, the operators $P_+=\sqrt{-1/\alpha}\,a_m^+$, $P_-=\sqrt{-1/\alpha}\,a_m$ and the unit operator I form a basis of $\mathbb{L}_m$ such that

$$[P_+,P_-]=-I \qquad [I,P_\pm]=0$$

that is, $\mathbb{L}_m$ is isomorphic to the Heisenberg-Weyl algebra $h(2)$. If $\sigma(s)=1-s^2$ then $K_+=a_m^+$, $K_-=a_m$ and $K_0=R_m$ form a basis of $\mathbb{L}_m$ such that

$$[K_+,K_-]=-2K_0 \qquad [K_0,K_\pm]=\pm K_\pm. \qquad \blacksquare$$

The operator a_m can be regarded as an annihilation operator, and a_m^+ as a creation operator.

4.2 Coherent states

Let $m\in\mathbb{N}$ be a fixed natural number. The functions $|0\rangle$, $|1\rangle$, $|2\rangle$, $\cdots$, where

$$|n\rangle=\phi_{m+n,m} \tag{30}$$

satisfy the relations

$$
\begin{aligned}
a_m|n\rangle &= \begin{cases} 0 & \text{if} \quad n=0 \\ \sqrt{e_n}\,\mathrm{e}^{\mathrm{i}\gamma(e_n-e_{n-1})}|n-1\rangle & \text{if} \quad n>0 \end{cases} \\
a_m^+|n\rangle &= \sqrt{e_{n+1}}\,\mathrm{e}^{-\mathrm{i}\gamma(e_{n+1}-e_n)}|n+1\rangle \\
(H_m-\lambda_m)|n\rangle &= e_n|n\rangle
\end{aligned} \tag{31}
$$

where

$$
\begin{aligned}
e_n &= \lambda_{m+n}-\lambda_m \\
&= \begin{cases} -\alpha n & \text{if} \quad \sigma(s)\in\{1,s\} \\ n(n+2m-\alpha-1) & \text{if} \quad \sigma(s)=1-s^2. \end{cases}
\end{aligned} \tag{32}
$$

For each $z\in\mathbb{C}$, the function

$$|z,\gamma\rangle=\sum_{n=0}^{\infty}\frac{z^n\,\mathrm{e}^{-\mathrm{i}\gamma e_n}}{\sqrt{\varepsilon_n}}|n\rangle \tag{33}$$

with

$$\varepsilon_n=\begin{cases} 1 & \text{if} \quad n=0 \\ e_1e_2...e_n & \text{if} \quad n>0\,. \end{cases} \tag{34}$$

is an eigenfunction of a_m

$$a_m|z,\gamma\rangle=z|z,\gamma\rangle \tag{35}$$

and

$$\langle z,\gamma|z,\gamma\rangle=\begin{cases} \mathrm{e}^{-\frac{|z|^2}{\alpha}} & \text{if } \sigma(s)\in\{1,\ s\} \\ {}_0F_1(2m-\alpha;|z|^2) & \text{if } \sigma(s)=1-s^2. \end{cases} \tag{36}$$

where

$$ {}_0F_1(c;z)=1+\frac{1}{c}\frac{z}{1!}+\frac{1}{c(c+1)}\frac{z^2}{2!}+\frac{1}{c(c+1)(c+2)}\frac{z^3}{3!}+\cdots \tag{37}$$

is the confluent hypergeometric function.
By using the notation $z=re^{\mathrm{i}\theta}$ and the modified Bessel function

$$K_\nu(z)=\frac{\pi}{2}\frac{I_{-\nu}(z)-I_\nu(z)}{\sin(\nu\pi)} \tag{38}$$

where

$$I_\nu(z)=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}z\right)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)} \tag{39}$$

we prove (following [2]) that $\{\ |z,\gamma\rangle\ |\ z\in\mathbb{C}\}$ is a system of coherent states.

Theorem 2. *The system of functions*

$$\{\ |z,\gamma\rangle\ |\ z\in\mathbb{C}\}$$

satisfies the resolution of identity

$$\int_{\mathbb{C}} d\mu\,|z,\gamma\rangle\langle z,\gamma|=I \tag{40}$$

for $d\mu$ defined as

$$d\mu=\begin{cases}\frac{-1}{\pi\alpha}e^{\frac{1}{\alpha}|z|^2}d(\mathrm{Re}\,z)\,d(\mathrm{Im}\,z)\\ \qquad\qquad for\ \sigma(s)\in\{1,s\}\\ \frac{2r^{2m-\alpha}}{\pi\Gamma(2m-\alpha)}K_{\frac{\alpha+1}{2}-m}(2r)\,dr\,d\theta\\ \qquad\qquad for\ \sigma(s)=1-s^2.\end{cases} \tag{41}$$

Proof. If $\sigma(s)\in\{1,s\}$ then

$$|z,\gamma\rangle=\sum_{n=0}^{\infty}\frac{z^n\,e^{-\mathrm{i}\gamma e_n}}{\sqrt{n!\,(-\alpha)^n}}|n\rangle. \tag{42}$$

By denoting $t=-\frac{r^2}{\alpha}$ and using the integration by parts we get

$$\frac{-1}{\pi\alpha}\int_{\mathbb{C}}d(\mathrm{Re}\,z)\,d(\mathrm{Im}\,z)|z,\gamma\rangle\langle z,\gamma|$$

$$=\frac{-1}{\pi\alpha}\sum_{n,n'}e^{-\mathrm{i}\gamma(e_n-e_{n'})}\times$$

$$\times\left(\int_0^\infty e^{\frac{r^2}{\alpha}}\frac{r^{n+n'+1}}{\sqrt{n!\,n'!\,(-\alpha)^{n+n'}}}dr\int_0^{2\pi}e^{\mathrm{i}(n-n')\theta}d\theta\right)|n\rangle\langle n'|$$

$$=\frac{-2}{\alpha}\sum_n\left(\int_0^\infty e^{\frac{r^2}{\alpha}}\frac{1}{n!}\left(\frac{r^2}{-\alpha}\right)^n r\,dr\right)|n\rangle\langle n|$$

$$=\sum_n\left(\int_0^\infty e^{-t}\frac{t^n}{n!}\,dt\right)|n\rangle\langle n|=\sum_n|n\rangle\langle n|=I.$$

If $\sigma(s)=1-s^2$ then

$$|z,\gamma\rangle=\sqrt{\Gamma(2m-\alpha)}\sum_{n=0}^{\infty}\frac{z^n\,e^{-\mathrm{i}\gamma e_n}}{\sqrt{n!\,\Gamma(n+2m-\alpha)}}|n\rangle \tag{43}$$

Denoting $d\mu=\mu(r)\,dr\,d\theta$ we get

$$\int_{\mathbb{C}}d\mu\,|z,\gamma\rangle\langle z,\gamma|$$

$$=\sum_{n=0}^{\infty}\frac{2\pi\Gamma(2m-\alpha)}{n!\,\Gamma(n+2m-\alpha)}\left(\int_0^\infty r^{2n}\mu(r)\,dr\right)|n\rangle\langle n|$$

and hence, we must have the relation (Mellin transformation)

$$2\pi\Gamma(2m-\alpha)\int_0^\infty r^{2n}\mu(r)\,dr=\Gamma(n+1)\,\Gamma(n+2m-\alpha). \tag{44}$$

The formula [4]

$$\int_0^\infty 2x^{\eta+\xi}K_{\eta-\xi}(2\sqrt{x})\,x^{n-1}dx=\Gamma(2\eta+n)\,\Gamma(2\xi+n)$$

for $x=r^2$, $\eta=\frac{1}{2}$, $\xi=m-\frac{\alpha}{2}$ becomes

$$4\int_0^\infty r^{2n}K_{\frac{\alpha+1}{2}-m}(2r)\,r^{2m-\alpha}dr=\Gamma(n+1)\Gamma(n+2m-\alpha). \tag{45}$$

The relations (44) and (45) lead to (41). ∎

If we consider the 'number' operator [1, 8]

$$N:\mathcal{H}\longrightarrow\mathcal{H}\qquad N|n\rangle=n|n\rangle \tag{46}$$

that is,

$$N=\sum_{n=0}^{\infty}n\,|n\rangle\langle n| \tag{47}$$

then the operator $H=H_m-\lambda_m$ can be written as

$$H=\sum_{n=0}^{\infty}e_n|n\rangle\langle n|$$

$$=\begin{cases}-\alpha N & \text{if}\quad \sigma(s)\in\{1,\ s\}\\ N(N+2m-\alpha-1) & \text{if}\quad \sigma(s)=1-s^2.\end{cases}$$

The operators a_m and $a_m^\perp$, where [8]

$$a_m^\perp=\frac{N}{H}a_m^+=\begin{cases}-\frac{1}{\alpha}a_m^+ & \text{if}\quad \sigma(s)\in\{1,\ s\}\\ \frac{1}{N+2m-\alpha-1}a_m^+ & \text{if}\quad \sigma(s)=1-s^2.\end{cases}$$

satisfy the relations

$$[a_m, a_m^{\perp}] = I \qquad [N, a_m^{\perp}] = a_m^{\perp} \qquad [N, a_m] = -a_m.$$

Therefore, we can consider the non-unitary displacement operator [8]

$$\begin{aligned} D(z) &= \exp(z\, a_m^{\perp} - \overline{z}\, a_m) \\ &= \exp\left(-\tfrac{1}{2}|z|^2\right)\ \exp(z\, a_m^{\perp})\ \exp(-\overline{z}\, a_m) \end{aligned}$$

and

$$|z,\gamma\rangle = D(z)\,|0\rangle \qquad \text{for any } z \in \mathbb{C}. \tag{48}$$

Since the Hermitian operators

$$X = \frac{1}{\sqrt{2}}(a_m^+ + a_m) \qquad P = \frac{\mathrm{i}}{\sqrt{2}}(a_m^+ - a_m) \tag{49}$$

satisfy the commutation relation

$$[X, P] = \mathrm{i}[a_m, a_m^+] \tag{50}$$

and $|z,\gamma\rangle$ are eigenstates of a_m, the coherent states $|z,\gamma\rangle$ minimize the uncertainty relation [8]

$$(\Delta X)^2 (\Delta P)^2 \geq \frac{1}{4}\langle \mathrm{i}[X, P]\rangle^2. \tag{51}$$

The presence of the phase factor in definition of $|z,\gamma\rangle$ leads to the temporal stability of these coherent states

$$\mathrm{e}^{-\mathrm{i}tH}|z,\gamma\rangle = |z,\gamma + t\rangle. \tag{52}$$

4.3 Analytical representations

The space

$$\mathcal{F}_m = \left\{ f : \mathbb{C} \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is an analytic function} \\ \int_{\mathbb{C}} |f(z)|^2\, d\mu < \infty \end{array} \right\}$$

is a Hilbert space with the inner product

$$(f, g) = \int_{\mathbb{C}} \overline{f(z)}\, g(z)\, d\mu \tag{53}$$

where $d\mu$ is the measure defined by (41).
Following Bargmann [3], we associate to each $\varphi : (a, b) \longrightarrow \mathbb{C}$ from $\mathcal{H}$

$$\varphi = \sum_{n=0}^{\infty} c_n |n\rangle \qquad \text{with} \qquad \sum_{n=0}^{\infty} |c_n|^2 < \infty \tag{54}$$

the entire function $f : \mathbb{C} \longrightarrow \mathbb{C}$

$$f(z) = \langle \overline{z}, \gamma|\varphi\rangle = \sum_{n=0}^{\infty} \frac{z^n \mathrm{e}^{\mathrm{i}\gamma e_n}}{\sqrt{\varepsilon_n}} \langle n|\varphi\rangle = \sum_{n=0}^{\infty} \frac{z^n \mathrm{e}^{\mathrm{i}\gamma e_n}}{\sqrt{\varepsilon_n}} c_n. \tag{55}$$

Particularly, the function

$$u_n : \mathbb{C} \longrightarrow \mathbb{C} \qquad u_n(z) = \frac{z^n \mathrm{e}^{\mathrm{i}\gamma e_n}}{\sqrt{\varepsilon_n}} \tag{56}$$

corresponds to $\varphi = |n\rangle$, for any $n \in \mathbb{N}$.

From the relation (44) we get

$$(z^n, z^k) = \varepsilon_n\, \delta_{nk} = \begin{cases} \varepsilon_n & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases} \tag{57}$$

whence

$$(f, g) = \sum_{n=0}^{\infty} \varepsilon_n\, \overline{c_n}\, d_n \tag{58}$$

for any two elements $f, g \in \mathcal{F}_m$

$$f = \sum_{n=0}^{\infty} c_n z^n \quad \text{and} \quad g = \sum_{n=0}^{\infty} d_n z^n.$$

For an entire function $f = \sum_{n=0}^{\infty} c_n z^n$ we have

$$f \in \mathcal{F}_m \quad \Longleftrightarrow \quad \sum_{n=0}^{\infty} \varepsilon_n\, |c_n|^2 < \infty. \tag{59}$$

Since

$$(u_n, f) = \sqrt{\varepsilon_n}\, \mathrm{e}^{-\mathrm{i}\gamma e_n} c_n \tag{60}$$

the orthonormal system u_0, u_1, u_2, ... is complete

$$||f||^2 = \sum_{n=0}^{\infty} \varepsilon_n\, |c_n|^2 = \sum_{n=0}^{\infty} |(u_n, f)|^2. \tag{61}$$

The isometry

$$\mathcal{H} \longrightarrow \mathcal{F}_m : \quad \sum_{n=0}^{\infty} c_n |n\rangle \mapsto \sum_{n=0}^{\infty} c_n\, u_n$$

allows us to identify the two Hilbert spaces, and to get

$$\begin{aligned} a_m\, u_n &= \begin{cases} 0 & \text{if } \ n = 0 \\ \sqrt{e_n}\, \mathrm{e}^{\mathrm{i}\gamma(e_n - e_{n-1})} u_{n-1} & \text{if } \ n > 0 \end{cases} \\ a_m^+\, u_n &= \sqrt{e_{n+1}}\, \mathrm{e}^{-\mathrm{i}\gamma(e_{n+1} - e_n)} u_{n+1} \\ R_m\, u_n &= (e_n - e_{n+1}) u_n \end{aligned} \tag{62}$$

whence

$$\begin{aligned} a_m\, z^n &= e_n\, z^{n-1} \\ a_m^+\, z^n &= z^{n+1} \\ R_m\, z^n &= (e_n - e_{n+1}) z^n. \end{aligned} \tag{63}$$

From the relation

$$e_n = \begin{cases} -\alpha n & \text{if} \quad \sigma(s) \in \{1, s\} \\ n(n + 2m - \alpha - 1) & \text{if} \quad \sigma(s) = 1 - s^2. \end{cases} \tag{64}$$

one gets

$$a_m = -\alpha \frac{d}{dz} \qquad a_m^+ = z \qquad R_m = \alpha \tag{65}$$

in the case $\sigma(s) \in \{1, s\}$, and

$$\begin{aligned} a_m &= z\tfrac{d^2}{dz^2} + (2m - \alpha)\tfrac{d}{dz} \\ a_m^+ &= z \\ R_m &= -2z\tfrac{d}{dz} - 2m + \alpha \end{aligned} \tag{66}$$

in the case $\sigma(s) = 1 - s^2$.

5 Coherent states in the case $\sigma(s) \in \{s^2 - 1,\, s^2,\, s^2 + 1\}$

Let $\tau(s) = \alpha s + \beta$ be a fixed polynomial,

$$L = \max\{l \in \mathbb{N} \mid l < (1 - \alpha)/2\} \tag{67}$$

and let m be a fixed natural number with $0 \leq m \leq L$. The functions $|0\rangle$, $|1\rangle$, ..., $|\mathcal{L}\rangle$, where $\mathcal{L} = L - m$ and

$$|n\rangle = \phi_{m+n,m} \tag{68}$$

are orthogonal and span a $(\mathcal{L} + 1)$-dimensional space

$$\mathcal{E}_m = \text{span}\{|0\rangle, |1\rangle, ..., |\mathcal{L}\rangle\}. \tag{69}$$

5.1 Creation and annihilation operators

By following [16] and the analogy with the case $\sigma(s) \in \{1,\, s,\, 1 - s^2\}$ we define the creation and annihilation operators $\tilde{a}_m^+$, $\tilde{a}_m : \mathcal{E}_m \longrightarrow \mathcal{E}_m$

$$\begin{aligned} \tilde{a}_m|n\rangle &= \begin{cases} 0 & \text{if} \quad n = 0 \\ \sqrt{\tilde{e}_n}\, \mathrm{e}^{\mathrm{i}\gamma(e_n - e_{n-1})}|n - 1\rangle & \text{if} \quad n > 0 \end{cases} \\ \tilde{a}_m^+|n\rangle &= \begin{cases} \sqrt{\tilde{e}_{n+1}}\, \mathrm{e}^{-\mathrm{i}\gamma(e_{n+1} - e_n)}|n + 1\rangle & \text{if} \quad n < \mathcal{L} \\ 0 & \text{if} \quad n = \mathcal{L} \end{cases} \end{aligned} \tag{70}$$

where $\tilde{e}_n = n(\mathcal{L} - n + 1)$ and

$$e_n = \lambda_{m+n} - \lambda_m = -n(n + 2m + \alpha - 1). \tag{71}$$

These operators satisfy the relations

$$\begin{aligned} [\tilde{a}_m^+, \tilde{a}_m] &= 2\, \tilde{R}_m \\ [\tilde{R}_m, \tilde{a}_m^+] &= \tilde{a}_m^+ \\ [\tilde{R}_m, \tilde{a}_m] &= -\tilde{a}_m \end{aligned} \tag{72}$$

where $\tilde{R}_m$ is the operator $\tilde{R}_m = N - \frac{\mathcal{L}}{2}$ defined by using the 'number operator'

$$N : \mathcal{E}_m \longrightarrow \mathcal{E}_m \qquad N|n\rangle = n|n\rangle. \tag{73}$$

Therefore, the Lie algebra generated by $\tilde{a}_m^+$ and $\tilde{a}_m$ is isomorphic to $su(2)$. The functions $|n\rangle$ are eigenfunctions of the operator $H = H_m - \lambda_m$

$$H|n\rangle = e_n|n\rangle. \tag{74}$$

5.2 Coherent states

By following [16] and the analogy with the case $\sigma(s) \in \{1,\, s,\, 1 - s^2\}$, we consider for each $z \in \mathbb{C}$ the function

$$|z, \gamma\rangle = \sum_{n=0}^{\mathcal{L}} \frac{z^n\, \mathrm{e}^{-\mathrm{i}\gamma e_n}}{\sqrt{\tilde{\varepsilon}_n}}|n\rangle \tag{75}$$

where

$$\tilde{\varepsilon}_n = \begin{cases} 1 & \text{if} \quad n = 0 \\ \tilde{e}_1 \tilde{e}_2 ... \tilde{e}_n & \text{if} \quad n > 0\,. \end{cases} \tag{76}$$

The system $\{\ |z, \gamma\rangle \mid z \in \mathbb{C}\}$ is an overcomplete system of functions in the finite dimensional space $\mathcal{E}_m$ with

$$\langle z, \gamma | z, \gamma\rangle = \sum_{n=0}^{\mathcal{L}} \frac{|z|^{2n}}{\tilde{\varepsilon}_n} \tag{77}$$

and the property (temporal stability)

$$\mathrm{e}^{-\mathrm{i}tH}|z, \gamma\rangle = |z, \gamma + t\rangle. \tag{78}$$

It can be regarded as a system of coherent states in $\mathcal{E}_m$.

7 Concluding remarks

The associated hypergeometric-type functions can be studied together in a unified formalism, and are directly related to the bound-state eigenfunctions of some important Schrödinger equations (Pöschl-Teller, Morse, Scarf, etc.).

It is useful to obtain fundamental versions (at the level of associated special functions) for some methods and formulae from quantum mechanics because in this way one can extend results known in particular cases to other quantum systems. A large number of formulae occurring in various applications of quantum mechanics follow from a very small number of fundamental mathematical results.

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